1 Area calculations

1.1 Area of an ellipse or a part of it

1.1.1 Without using parametric equations

We calculate the area in the first quadrant. We start from the standard equation of the ellipse and we put that equation in the form y = f (x). In the first quadrant, x and y are positive.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Leftrightarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$$

The area is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Leftrightarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$$

We first calculate a primitive function

$$I = \int \sqrt{a^2 - x^2} dx$$

We use integration by parts

$$u = \sqrt{a^2 - x^2} \qquad dv = dx$$

$$\begin{aligned} I &= \sqrt{a^2 - x^2}\ \ x - \int x \frac{-x}{\sqrt{a^2 - x^2}} dx \\ &= \sqrt{a^2 - x^2}\ \ x - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}} dx \\ &= \sqrt{a^2 - x^2}\ \ x - \int \sqrt{a^2 - x^2} dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} \\ 2I &= \sqrt{a^2 - x^2}\ \ x + a^2 \arcsin\frac{x}{a} \end{aligned}$$

The area is:

$$\begin{aligned} A &= \frac{b}{a}\left[\frac{1}{2}\sqrt{a^2 - x^2}\ \ x + \frac{a^2}{2}\arcsin\frac{x}{a}\right]_0^a \\ &= ab\pi/4 \end{aligned}$$

The area of the ellipse is πab.

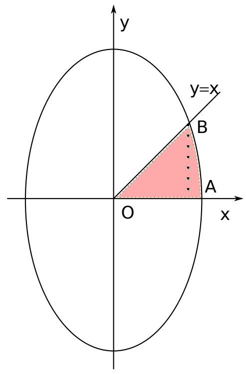
1.1.2 Using parametric equations

The parametric equations of an ellipse are $x = a\cos t$ and $x = b\sin t$.

We calculate the area in the first quadrant. t ranges from $\pi/2$ to 0.

$$A = \int_0^a y dx = -\int_{\pi/2}^0 b\sin t . a\sin t dt = -\frac{1}{2}ab\int_{\pi/2}^0 (1 - \cos 2t) dt = \ \ \ldots \ \ = ab\pi/4$$

The area of the ellipse is πab.



1.1.3 Area of a part of an ellipse

We calculate the area of the part of the ellipse $x^2+y^2/3=1$ in the first quadrant and enclosed by the x-axis and the line $y=x$. We work with the parametric equations $x=\cos t \;\; y=\sqrt{3}\sin t$ We first calculate the t-value corresponding to point B. This is a point on the ellipse where the abscis equals the ordinate.

$$\begin{aligned}\cos t &= \sqrt{3}\sin t\\ \tan t &= \frac{1}{\sqrt{3}}\\ t &= \pi/6\end{aligned}$$

The area of the part of the ellipse on the right of the dotted line is

$$\begin{aligned}A &= \int_{\frac{\sqrt{3}}{2}}^{1} y dx = \int_0^{\pi/6} \sqrt{3}\sin t . \sin t dt\\ &= \frac{\sqrt{3}}{2}\int_0^{\pi/6}(1-\cos 2t)dt = \quad \ldots \quad = \frac{\pi\sqrt{3}}{12} - \frac{3}{8}\end{aligned}$$

The area of the triangle is $\frac{1}{2}.\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2} = \frac{3}{8}$. The requested area is $\frac{\pi\sqrt{3}}{12}$.

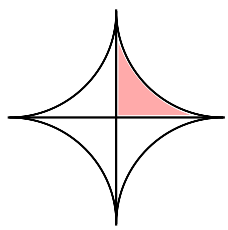
1.2 Area under a cycloid arc

The parametric equations of a cycloid are
$x = r(t-\sin t) \;\; y = r(1-\cos(t))$.
t is in $[0,2\pi]$.

$$\begin{aligned}A &= r^2\int_0^{2\pi}(1-\cos t)(1-\cos t)dt\\ &= r^2\int_0^{2\pi}(1-2\cos t+\cos^2 t)dt\\ &= r^2\int_0^{2\pi}(1-2\cos t+\frac{1}{2}+\frac{\cos 2t}{2})dt\\ &= \ldots\\ &= 3\pi r^2\end{aligned}$$

1.3 Area astroid

The parametric equations are $x = r\cos^3 t \;\; y = r\sin^3 t$. We calculate first a quarter of the surface.



$$\begin{aligned}A &= r^2\int_{\pi/2}^{0}\sin^3 t \;\; 3\cos^2 t \;\; (-\sin t)dt\\ &= -3r^2\int_{\pi/2}^{0}\cos^2 t\sin^4 t dt\end{aligned}$$

Now is

$$\begin{aligned}\sin^4 t\cos^2 t &= (1/4)(2\sin t\cos t)^2.(1/2)2\sin^2 t\\ &= (1/4)\sin^2 2t.(1/2)(1-\cos 2t)\\ &= (1/8)\sin^2 2t - (1/8)\sin^2 2t\cos 2t\end{aligned}$$

Hence

$$\begin{aligned}\int \cos^2 t \sin^4 t dt &= (1/8)\int \sin^2 2t dt - (1/8)\int \sin^2 2t \cos 2t dt\\ &= (1/16)\int (1-\cos 4t)dt - (1/16)\int \sin^2 2t d \sin 2t\\ &= \frac{t}{16} - \frac{1}{64}\sin(4t) - \frac{1}{48} sin^3 2t + C\end{aligned}$$

The area of the astroid is $(3/8)\pi r^2$.

1.4 Difference of two areas

The graph of $y = e^{-x/10}\sin x$ is a damped oscillation. Plot the graph. We calculate the difference in area between the parts above the x-axis and the parts below the x-axis in $[0, \infty]$.

First, we calculate

$$I = \int e^{\frac{-x}{10}} \sin x dx$$

Using integration by parts.

$$\begin{aligned}I &= -e^{-x/10} - \frac{1}{10}\int e^{-x/10}\cos x dx\\ &= -e^{-x/10}\cos x - \frac{1}{10}\left[e^{-x/10}\sin x + \frac{1}{10}\int e^{-x/10}\sin x dx\right]\\ &= -e^{-x/10}\cos x - \frac{1}{10}e^{\frac{x}{10}}\sin x - \frac{1}{100}\int e^{-x/10}\sin x dx\\ \frac{101}{100}I &= -e^{-x/10}(\cos x + \frac{1}{10}\sin x)\\ &= -\frac{1}{10}e^{-x/10}(10\cos x + \sin x)\\ I &= -\frac{10}{110}e^{-x/10}(10\cos x + \sin x)\end{aligned}$$

If we now take this between the boundaries 0 and ∞ then we find approximately 1. So, the difference in area between the parts above the x-axis and the parts below the x-axis, in $[0, \infty]$, is approximately 1.

1.5 Translation of a circle

Take only the part of the circle $x^2 + y^2 = r^2$ above the x-axis. We shift this arc down over a distance a until the area above the x-axis is halved. We want to find this special value of a.

The solution is simplified by noting that, in the shifted position, the area above the x-axis is the same as the area between the arc under the x-axis and the x-axis.

The equation of the shifted arc is

$$y = \sqrt{r^2 - x^2} - a$$

We calculate a such that

$$\int_{-r}^{r} (\sqrt{r^2 - x^2} - a)dx = 0$$

$$\int_{-r}^{r} \sqrt{r^2 - x^2} = [ax]_{-r}^{r}$$

But the left hand side is the expression for the area of a semicircle and this is equal to $\pi r^2/2$. The right hand side is 2ar. So $a = \pi.r/4$.

1.6 An appropriate upper limit

Determine t so that the area under the curve y = f(x), in the interval [ln (2),t], is equal to $\pi/6$. With

$$f(x) = \frac{1}{\sqrt{e^x - 1}}$$

We will require that

$$\int_{ln(2)}^{t} \frac{1}{\sqrt{e^x - 1}} dx = \pi/6$$

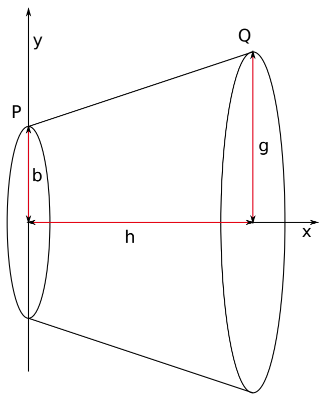
To calculate the indefinite integral we use the substitution $e^x - 1 = u^2$ with $u > 0$, then $e^x = u^2 + 1$ and $e^x dx = 2udu$. So

$$\begin{aligned}\int \frac{1}{\sqrt{e^x - 1}} dx &= \int \frac{2udu}{u(u^2+1)} \\ &= 2\int \frac{du}{u^2+1} \\ &= 2\,\mathrm{atan}(u) \\ &= 2\,\mathrm{atan}\sqrt{e^x - 1} + C\end{aligned}$$

The equation becomes

$$\begin{aligned} & \int_{ln(2)}^{t} \frac{1}{\sqrt{e^x - 1}} dx = \pi/6 \\ \Leftrightarrow \quad & \left[2\,\mathrm{atan}\sqrt{e^x - 1}\right]_{ln(2)}^{t} = \pi/6 \\ \Leftrightarrow \quad & 2\,\mathrm{atan}\sqrt{e^x - 1} - 2\,\mathrm{atan}(1) = \pi/6 \\ \Leftrightarrow \quad & 2\,\mathrm{atan}\sqrt{e^x - 1} = \pi/6 + \pi/2 \\ \Leftrightarrow \quad & \sqrt{e^x - 1} = \tan \pi/3 \\ \Leftrightarrow \quad & t = ln(4)\end{aligned}$$

2 Volume of a body of revolution



2.1 Volume of a truncated cone

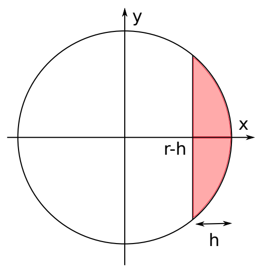
The radius of the base and upper surface are called, respectively, g and b. The height is h.
The truncated cone is created by turning of a rectangular trapezium.
The slope of PQ is (g-b)/h and the equation of PQ is $y = x(g - b)/h + b$.
We call G and B respectively, the area of base and upper surface.
The volume of a truncated cone is

$$
\begin{aligned}
I &= \pi \int_0^h (\frac{g-b}{h}x+b)^2 dx \\
&= \pi \frac{h}{g-b} \int_0^h (\frac{g-b}{h}x+b)^2 d(\frac{g-b}{h}x+b) \\
&= \pi \frac{h}{3(g-b)} \left[(\frac{g-b}{h}x+b)^3 \right]_0^h \\
&= \pi \frac{h}{3(g-b)} \left[(g-b+b)^3 - b^3 \right] \\
&= \frac{\pi h}{3} \frac{g^3-b^3}{g-b} \\
&= \frac{\pi h}{3} (g^2+gb+b^2) \\
&= \frac{h}{3} (\pi g^2 + \pi g b + \pi b^2) \\
&= \frac{h}{3} (\pi g^2 + \pi b^2 + \sqrt{\pi g^2 \pi b^2}) \\
&= \frac{h}{3} (G + B + \sqrt{GB})
\end{aligned}
$$

2.2 Volume of parts of a sphere

The sphere has radius r. The volume of the spherical cap with height h is

$$
\begin{aligned}
I &= \pi \int_{r-h}^{r} (r^2 - x^2) dx \\
&= \pi \left[r^2 x - \frac{x^3}{3} \right]_{r-h}^{r} \\
&= \dots \\
&= \frac{\pi}{3} h^2 (3r - h)
\end{aligned}
$$



Now, we make h equal to 2r, we have the volume $\frac{4}{3}\pi r^3$ of the sphere.

The volume of a segment can easily be calculated as the difference of two spherical caps.

2.3 Volume of a rotating arc of a cycloid

The parametric equations of a cycloid are
$x = r(t - \sin t)$ $y = r(1 - \cos(t))$.
t is in [0,2π].

The arc rotates around the x-axis in the interval$[0, 2\pi r]$.
With these boundaries correspond to the t-values 0 en 2π.

$$
\begin{aligned}
I &= \pi \int_0^{2\pi r} y^2 dx \\
&= \pi \int_0^{2\pi} r^2 (1-\cos t)^2 r (1-\cos t) dt \\
&= \pi r^3 \int_0^{2\pi} (1-\cos t)^3 dt
\end{aligned}
$$

We first calculate the indefinite integral.

$$
\begin{aligned}
I' &= \int (1-\cos t)^3 dt \\
&= \int (1 - 3\cos t + 3\cos^2 t - \cos^3 t)dt \\
&= t - 3\sin t + (3/2)\int (1+\cos 2t)dt - \int \cos^3 t dt \\
&\text{Say sin t = u} \\
&= t - 3\sin t + (3/2)t + (3/2)\int \cos 2t dt - u + u^3/3 \\
&= t - 3\sin t + (3/2)t + (3/4)\sin 2t - \sin t + (1/3)\sin^3 t + C
\end{aligned}
$$

If we then calculate I we find, after simplification, $5\pi^2 r^3$.

2.4 Volume of a rotating astroid

The parametric equations are $x = r\cos^3 t \;\; y = r\sin^3 t$. t changes from $\pi/2$ to 0. The volume is

$$
\begin{aligned}
I &= 2\pi \int_0^r y^2 dx \\
&= 2\pi \int_{\pi/2}^0 r^2 \sin^6 t . 3r(-\sin t)\cos^2 t dt \\
&= 6\pi r^3 \int_0^{\pi/2} \cos^2 t \sin^7 t dt
\end{aligned}
$$

To calculate the indefinite integral we set cos(t) = u. Then, the volume is easy to find.

3 Length of a curve

3.1 Example 1

Consider in [-1,1] the graph of the curve with equation

$$
y = \frac{2}{3}\sqrt{(1-x^2)^3} + 3
$$

We calculate the length of the curve.

$$
\begin{aligned}
y' &= 2x\sqrt{1+x^2} \\
1 + y'^2 &= 1 + 4x^2 + 4x^4 \\
\sqrt{1+y'^2} &= 1 + 2x^2
\end{aligned}
$$

$$
\begin{aligned}
L &= \int_{-1}^1 (1+2x^2)dx \\
&= \ldots \\
&= 10/3
\end{aligned}
$$

3.2 Example 2

Consider in [-1,1] the graph of the curve with equation

$$\frac{1}{2}(e^x + e^{-x})$$

We calculate the length of the curve.

$$\begin{aligned}
y' &= \frac{1}{2}(e^x - e^{-x}) \\
y'^2 &= \frac{1}{4}(e^{2x} - 2 + e^{-2x}) \\
1 + y'^2 &= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) \\
&= \frac{1}{4}(e^x + e^{-x})^2 \\
1 + y'^2 &= \frac{1}{2}(e^x + e^{-x}) \\
L &= \int_{-1}^{1} \frac{1}{2}(e^x + e^{-x})dx \\
&= \dots \\
&= e - \frac{1}{e}
\end{aligned}$$

3.3 Length of an arc of a cycloid

The parametric equations of a cycloid are
$x = r(t - \sin t)$ $y = r(1 - \cos(t))$.
t is in [0,2π].

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\sin t}{1 - \cos t} \\
(\frac{dy}{dx})^2 &= \frac{(\sin t)^2}{(1 - \cos t)^2} \\
1 + (\frac{dy}{dx})^2 &= \frac{2 - 2\cos t}{(1 - \cos t)^2} \\
&= \frac{2}{(1 - \cos t)}
\end{aligned}$$

$$\begin{aligned}
\sqrt{1 + (\frac{dy}{dx})^2} &= \sqrt{2}\frac{1}{\sqrt{1 - \cos t}} \\
L &= \sqrt{2}\int_0^{2\pi} \frac{1}{\sqrt{1 - \cos t}}(1 - \cos t)dt \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos t}dt \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{2sin^2\frac{t}{2}}dt \\
&= 2\int_0^{2\pi} sin\frac{t}{2}dt \\
&= \dots \\
&= 8
\end{aligned}$$

3.4 Length of an arc of an astroid

The parametric equations are $x = r\cos^3 t$ $y = r\sin^3 t$. t changes from $\pi/2$ to 0. We look first at one quarter of the required length.

$$\begin{aligned}
y' = \frac{dy}{dx} &= \frac{r.3\sin^2 t\cos t dt}{-r.3\cos^2 t\sin t dt}\\
&= -\frac{\sin t}{\cos t}\\
1 + y'^2 &= \frac{1}{\cos^2 t}\\
\sqrt{1 + y'^2} &= \frac{1}{\cos t}\\
dx &= -3r\cos^2 t\sin t dt
\end{aligned}$$

If dx is positive then dt is negative.

$$\begin{aligned}
L &= \int_{\pi/2}^{0} (-3r\cos t\sin t) dt\\
&= 3r \int_{0}^{\pi/2} \cos t\sin t dt\\
&= \dots\\
&= 3r/2
\end{aligned}$$

The full length is 6r.